Defensive Info Operations - Part I
Data Security & Cryptology

Hüseyin Hışıl

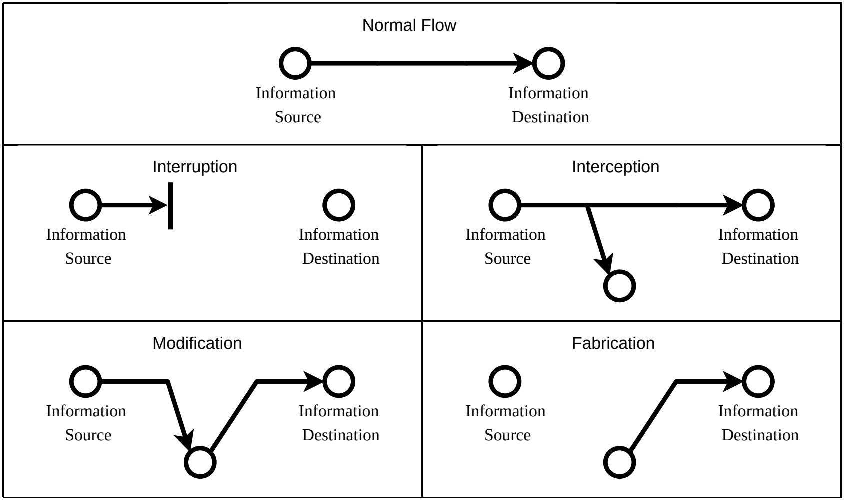
Computer Engineering Department
Engineering Faculty
Yaşar University

22 April 2012 / İzmir

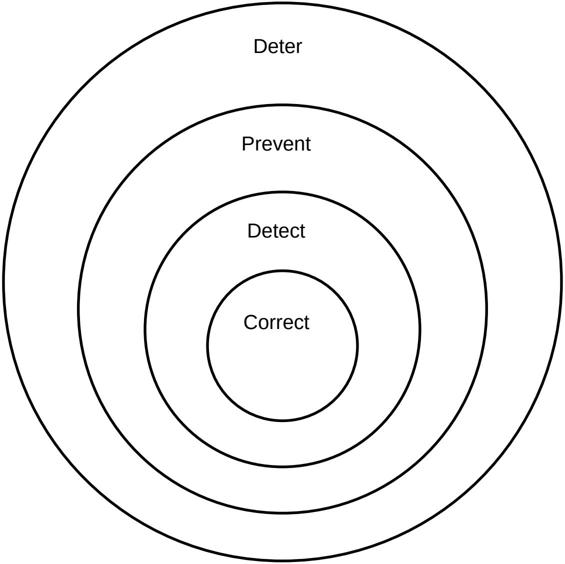
Acknowledgement: Some of the slides are compiled from Dr. Koltuksuz's lecture notes.

Possible Data Security Threats

Threat is a potential violation of security.



Layers of Defensive Data Security



Essential Services for Data Security & Network Security

- Availability:
  - Ensures that data remain to be sucessfully accessible. (Networking)
  - Interruption targets availability.
- Authentication:
  - Ensures that data really were sent by the claimed sender. (Cryptology)
  - Fabrication targets authentication.
- Confidentiality:
  - Ensures that data are accessed only by authorized parties. (Cryptology)
  - Interception targets confidentiality.
- Integrity:
  - Ensures that the original data is intact. (Coding Theory)
  - Modification targets integrity.

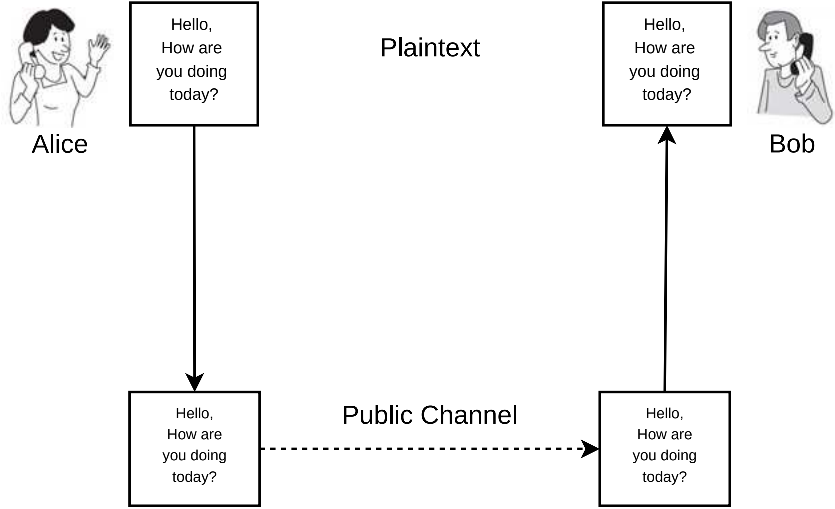
Note: Higher level services such as non-repudiation, access control, utility, possession, can be defined as needed.

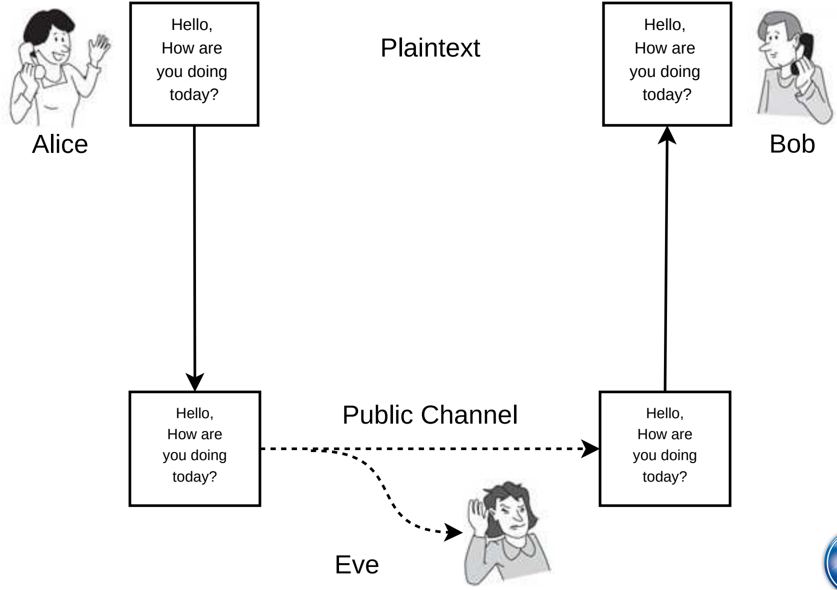
The World of Crypto

- Cryptography: The science of securing data.
- Cryptanalysis: The science of defeating cryptographic security.
- Coding theory: The science of converting the representation of data.
- Cryptology $=$ Cryptography $+$ Cryptanalysis $\pm$ Coding theory.
- $\Big($Cryptology$\Big) \sim \Big($

  (Logic) $\wedge$
  (Mathematics) $\wedge$
  (Computer science) $\wedge$
  (Computer engineering) $\wedge$
  (Electrical & Electronics engineering)

  $\Big)$.


Alice
Hello, How are you doing today?
Plaintext
Hello, How are you doing today?
Bob
Hello, How are you doing today?
Public Channel
Hello, How are you doing today?


Hello,
How are
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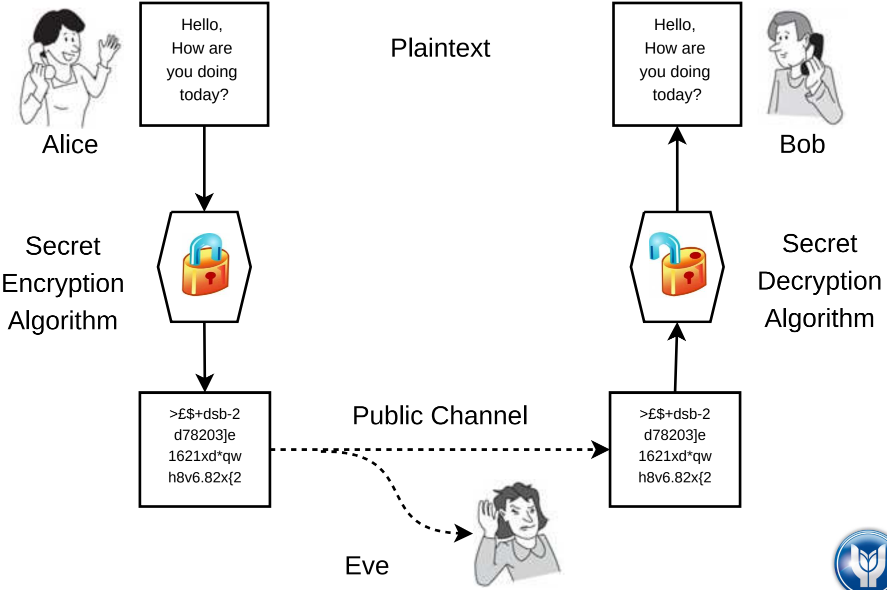
Plaintext
Hello,
How are
you doing
today?
Alice
Bob
Hello,
How are
you doing
today?
Public Channel
Hello,
How are
you doing
today?
Eve


Hello,
How are
you doing
today?
Plaintext
Hello,
How are
you doing
today?
Alice
Bob
Secret
Encryption
Algorithm
Secret
Decryption
Algorithm
>£$+dsb-2
d78203]e
1621xd*qw
h8v6.82x{2
Public Channel
>£$+dsb-2
d78203]e
1621xd*qw
h8v6.82x{2
Eve

The Archaic Ciphers - Selected Examples

- Ancient Greeks and Romans
  - 475 B.C. Spartans - Scytale Cipher.
  - 60 B.C. Julius Caesar - Substitution Cipher.
- Middle Ages
  - 1378 - 1417 Gabriele de Lavinde of Parma
- Renaissance
  - 1518 Johannes Trithemius: “Polygraphiae” (Steganographia), first printed work!
- 20th Century
  - 1917 Zimmermann Telegramme (codebooks)
  - 1926 Vernam, “one-time-pad”
  - 1939 - 1945 2nd World War: “Enigma” - “Purple”

Zimmermann Telegramme

WESTERN UNION TELEGRAM

NEWCOMB CARLTON, PRESIDENT

Send the following telegram, subject to the terms on back hereof, which are hereby agreed to

via Galveston

JAN 19 1917

GERMAN LEGATION

MEXICO CITY

130 13042 13401 8501 115 3528 416 17214 6491 11310
18147 18222 21560 10247 11518 23677 13605 3494 14936
98092 5905 11311 10392 10371 0302 21290 5161 39695
23571 17504 11269 18276 18101 0317 0228 17694 4473
23284 22200 19452 21589 67893 5569 13918 8958 12137
1333 4725 4458 5905 17166 13851 4458 17149 14471 6706
13850 12224 6929 14991 7382 15857 67893 14218 36477
5870 17553 67893 5870 5454 16102 15217 22801 17138
21001 17388 7446 23638 18222 6719 14331 15021 23845
3156 23552 22096 21604 4797 9497 22464 20855 4377
23610 18140 22260 5905 13347 20420 39689 13732 20667
6929 5275 18507 52262 1340 22049 13339 11265 22295
10439 14814 4178 6992 8784 7632 7357 6926 52262 11267
21100 21272 9346 9559 22464 15874 18502 18500 15857
2188 5376 7381 98092 16127 13486 9350 9220 76036 14219
5144 2831 17920 11347 17142 11264 7667 7762 15099 9110
10482 97556 3569 3670

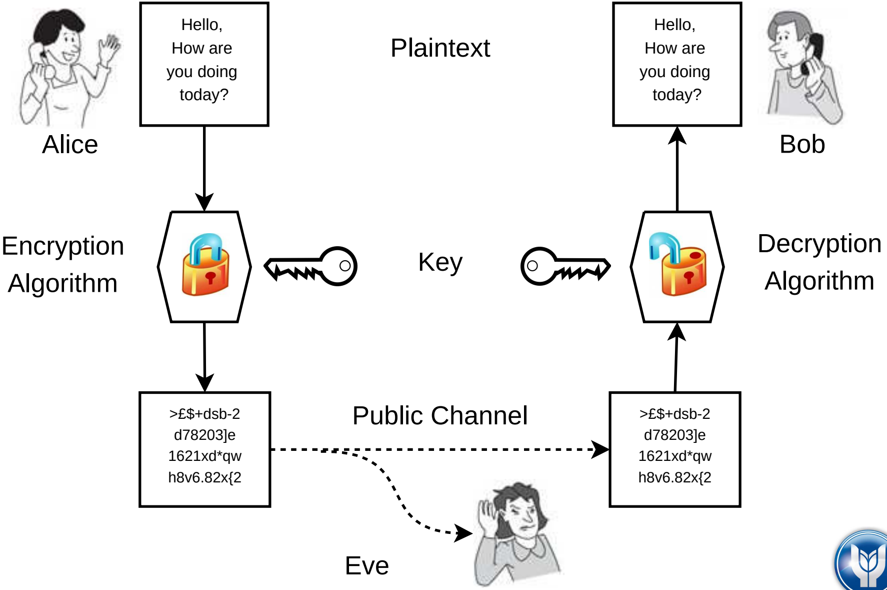
BERNSTORFF.

Charge German Embassy.

TELEGRAM RECEIVED.

FROM 2nd from London # 5747.

"We intend to begin on the first of February unrestricted submarine warfare. We shall endeavor in spite of this to keep the United States of America neutral. In the event of this not succeeding, we make Mexico a proposal of alliance on the following basis: make war together, make peace together, generous financial support and an understanding on our part that Mexico is to reconquer the lost territory in Texas, New Mexico, and Arizona. The settlement in detail is left to you. You will inform the President of the above most secretly as soon as the outbreak of war with the United States of America is certain and add the suggestion that he should, on his own initiative, invite Japan to immediate adherence and at the same time mediate between Japan and ourselves. Please call the President's attention to the fact that the ruthless employment of our submarines now offers the prospect of compelling England in a few months to make peace." Signed, ZIMMERMANN.


Hello,
How are
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Plaintext
Hello,
How are
you doing
today?
Alice
Bob
Encryption
Algorithm
Key
Decryption
Algorithm
>£$+dsb-2
d78203]e
1621xd*qw
h8v6.82x{2
Public Channel
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d78203]e
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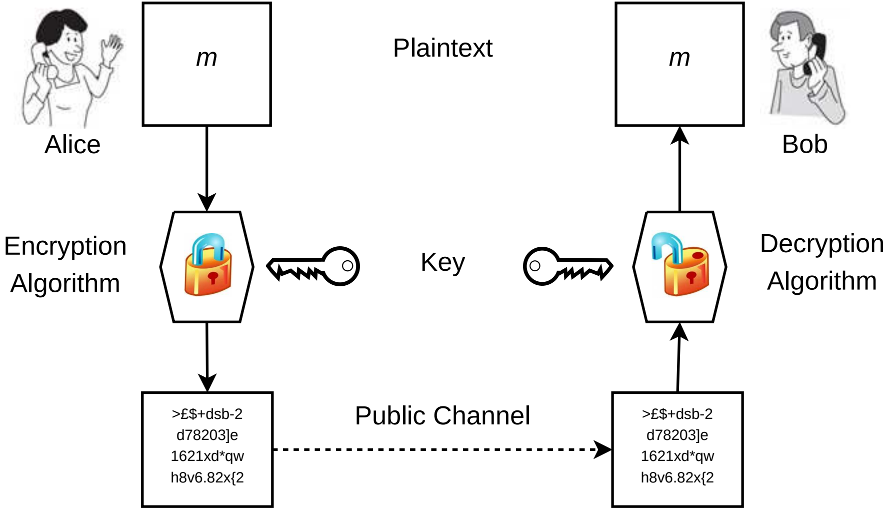
Eve


Alice
m
Plaintext
m
Bob
Encryption
Algorithm
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Algorithm
>£$+dsb-2
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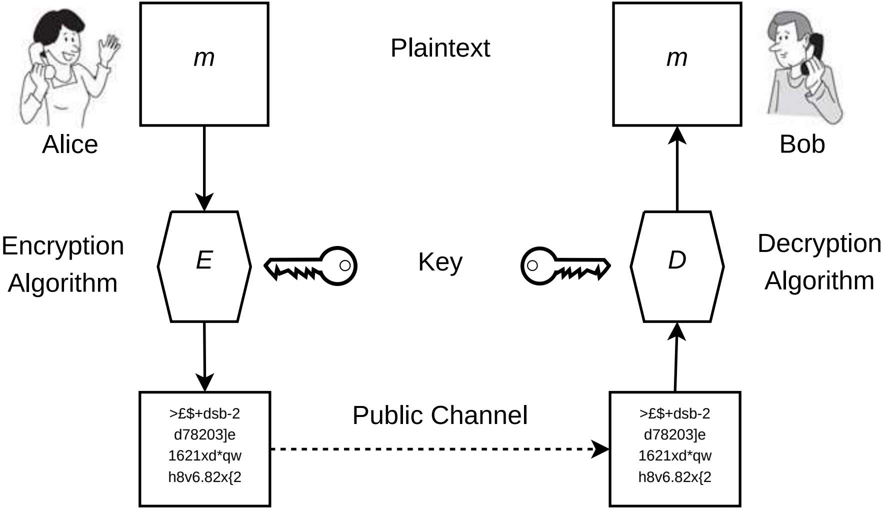

Plaintext
m
m
Alice
Bob
Encryption Algorithm
E
Key
D
Decryption Algorithm
Public Channel
>£$+dsb-2
d78203]e
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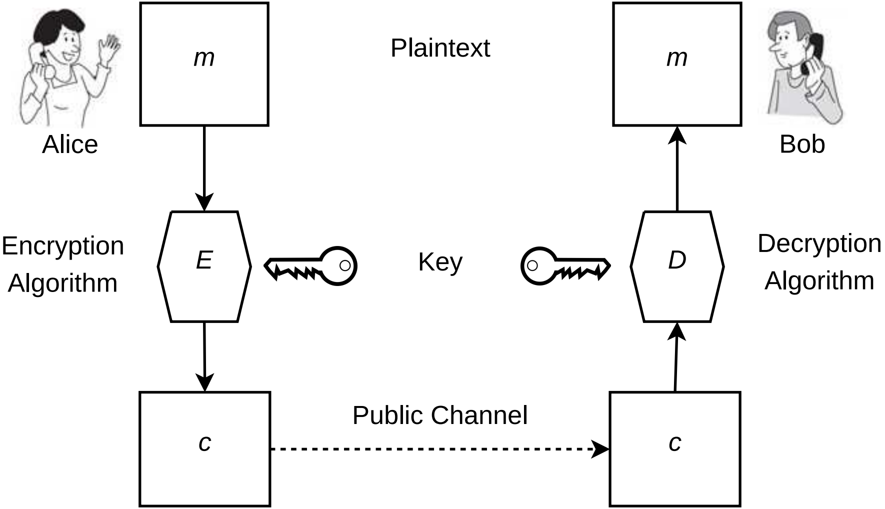
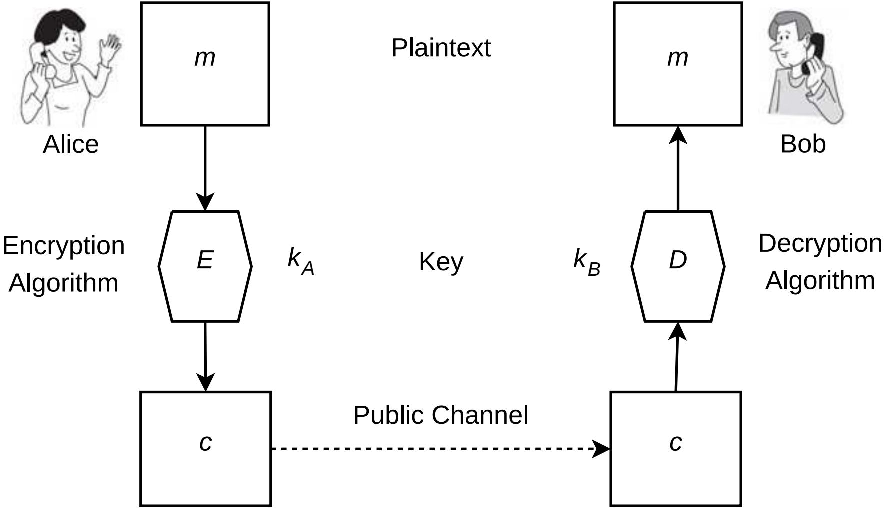
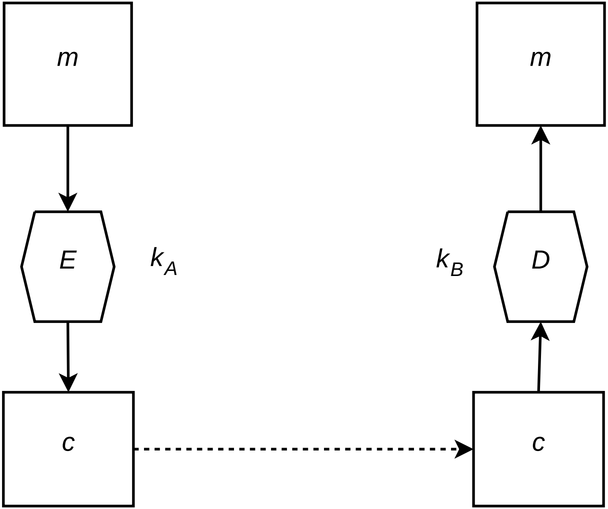
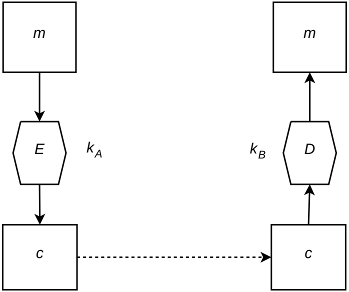

Alice
m
Plaintext
m
Bob
Encryption
Algorithm
E
Key
D
Decryption
Algorithm
c
Public Channel
c

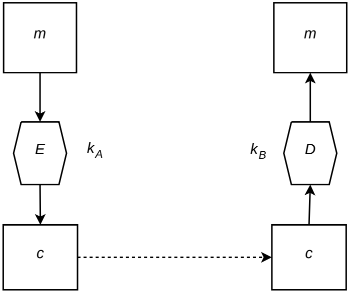


Alice
m
Plaintext
m
Bob
Encryption
Algorithm
E
k_A
Key
k_B
D
Decryption
Algorithm
c
Public Channel
c


m
m
E
k_A
k_B
D
c
c



- Obtaing the ciphertext: $E(m,k_A)=c$.
- Recovering the plaintext: $D(c,k_B)=m$.



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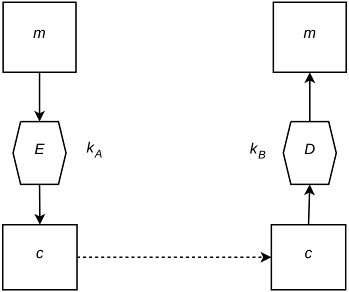
- Symmetric Key Cryptography: $k_A = k_B$.



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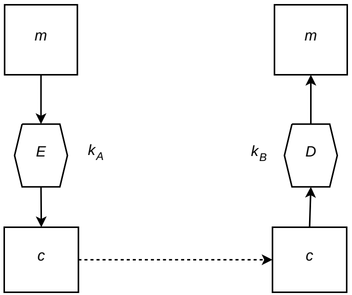

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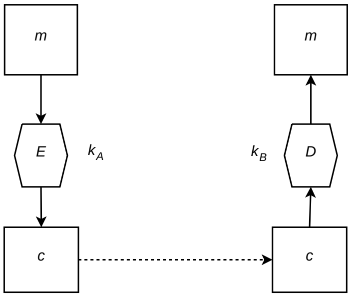

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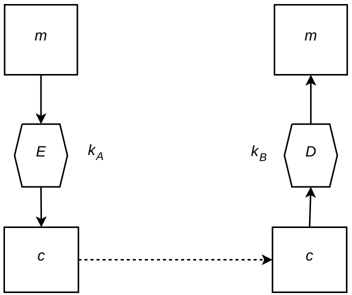

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- Symmetric Key Cryptography: $k_A = k_B$. (Shared / Secret Key Cry.)
- Asymmetric Key Cryptography: $k_A \neq k_B$. (Public Key Cry.)
- Cryptanalysis: Given c, E, D, find m.

Brute force attacks

Try all possible keys!

- 1-bit key: You have $2^2 = 2$ keys; one or the other
- 2-bit key: You have $2^4 = 4$ keys: one of the four
- 3-bit key: You have $2^3 = 8$ keys: one of the eight
- 4-bit key: You have $2^4 = 16$ keys: one of the sixteen
- ...
- 256-bit key: You have $2^{256} =$

  $$115792089237316195423570985008687907853269984665640564039457584007913129639936$$

  $$\approx 36717430630808027468154168254911183362909051454097083980041 09 \cdot 365 \cdot 24 \cdot 60 \cdot 60 \cdot 10^9$$

  keys!!!

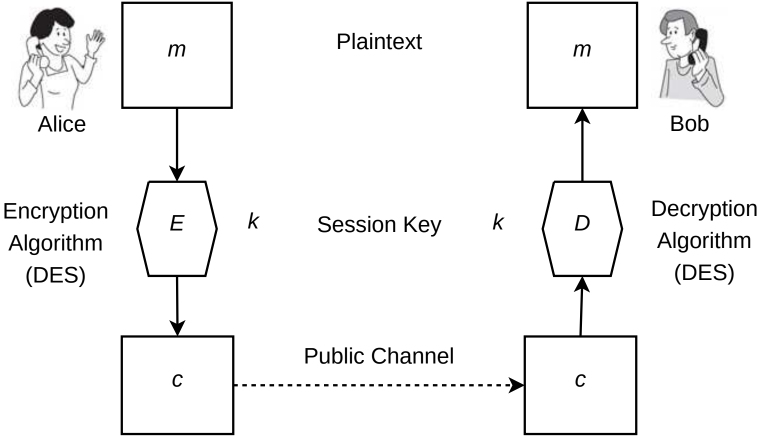
Contemporary Ciphers: Early Years

- 1971 IBM announces Lucifer, A Block cipher
- 1975 IBM offers Lucifer as a standard
- 1976 Diffie & Hellman, Public Key concept
- 1977 Lucifer gets approved by NIST as Data Encryption Standard (DES), a Block Cipher
- 1978 Rivest-Shamir-Adleman (RSA), Public Key Cryptosystem
- 1984 Shamir, Identity Based Cryptography
- 1985 Elliptic Curve Cryptograpgy (ECC)
- 1987 Stream cipher RC4
- 2001 Advanced Encryption Standard (AES)
- 2001 Boneh & Franklin, Identity Based Cryptography is feasible!

A Basic Taxonomy

- Symmetric systems
  - Block ciphers: DES, 3DES, IDEA, BLOWFISH, TWOFISH, AES, . . .
  - Stream ciphers: RC4, Dragon, HC-256, MICKEY, MOUSTIQUE, . . .
- Asymmetric systems:
  - Key exchange: DH
  - Encryption/Decryption: RSA, ELGAMAL, ECC, NTRU
  - Digital Signature: DSA, ECDSA

Data Encryption Standard (DES)



- Obtaing the ciphertext: $E(m,k) = c$.
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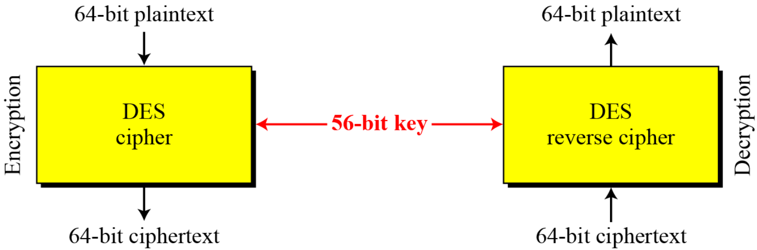
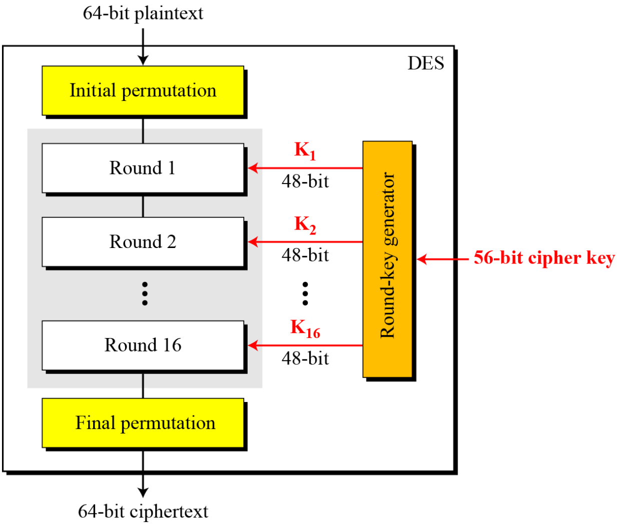
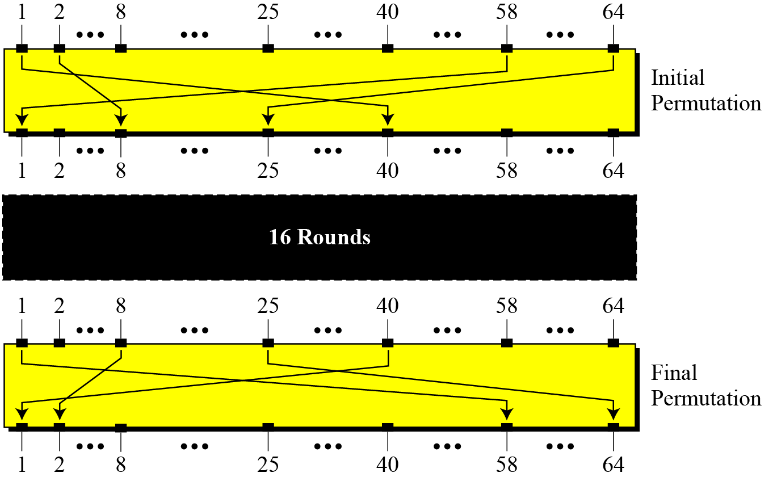
Data Encryption Standard (DES)



Data Encryption Standard (DES) / Rounds Overview



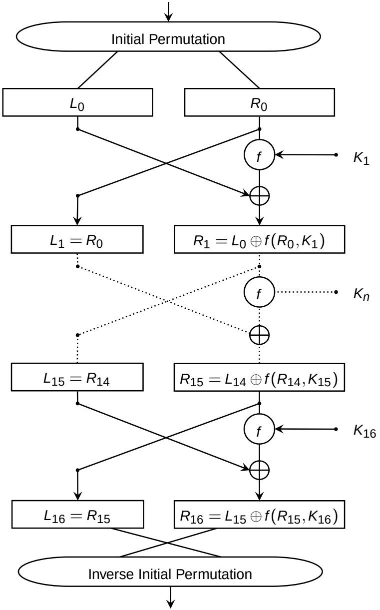
Data Encryption Standard (DES) / Initial & Final Permutation



Data Encryption Standard (DES) / Initial & Final Permutation

Initial Permutation	*Final Permutation*
58 50 42 34 26 18 10 02	40 08 48 16 56 24 64 32
60 52 44 36 28 20 12 04	39 07 47 15 55 23 63 31
62 54 46 38 30 22 14 06	38 06 46 14 54 22 62 30
64 56 48 40 32 24 16 08	37 05 45 13 53 21 61 29
57 49 41 33 25 17 09 01	36 04 44 12 52 20 60 28
59 51 43 35 27 19 11 03	35 03 43 11 51 19 59 27
61 53 45 37 29 21 13 05	34 02 42 10 50 18 58 26
63 55 47 39 31 23 15 07	33 01 41 09 49 17 57 25

Data Encryption Standard (DES) / Encryption & Decryption



We have

- $L_j = R_{j-1}$,
- $R_j = L_{j-1} \oplus f(R_{j-1}, k_j)$.

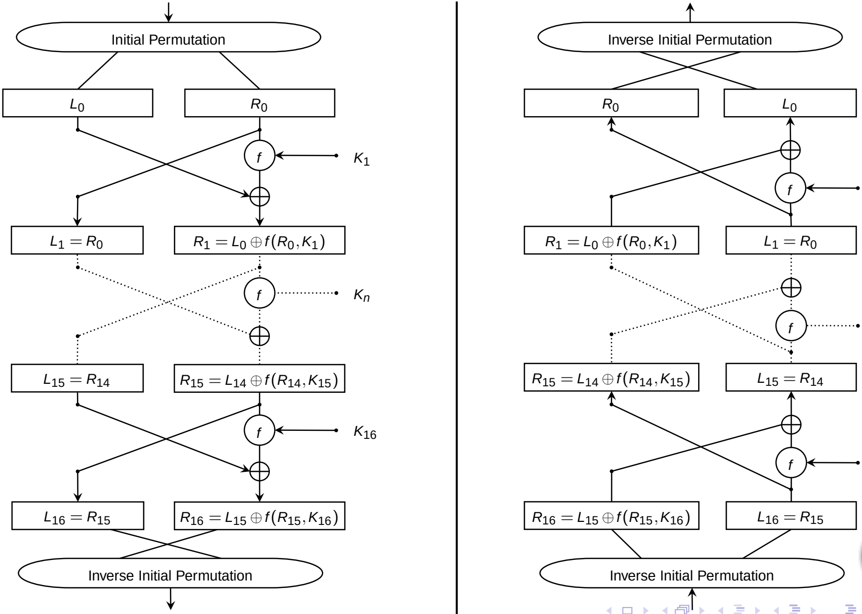
We can rewrite in the form

- $R_{j-1} = L_j$,
- $L_{j-1} = R_j \oplus f(R_{j-1}, k_j)$.

By substitution

- $L_{j-1} = R_j \oplus f(L_j, k_j)$.

Data Encryption Standard (DES) / Encryption & Decryption



Data Encryption Standard (DES) / The function $f(R_{i-1}, K_i)$

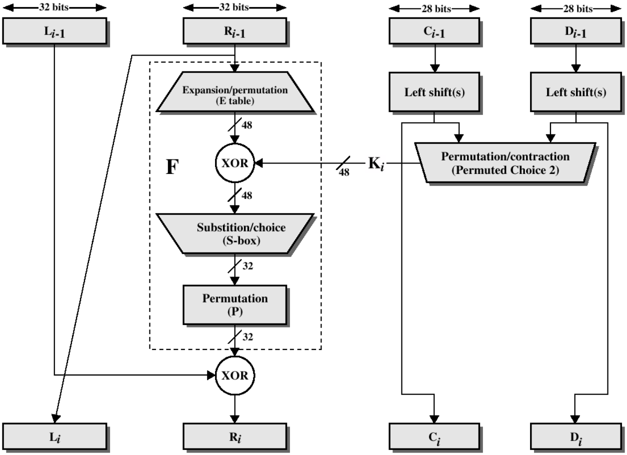


Figure 2.4 Single Round of DES Algorithm

EXPANSION PERMUTATION (32 -> 48): 32, 1, 2, 3, 4, 5, 4, 5, 6, 7, 8, 9, 8, 9, 10, 11, 12, 13, 12, 13, 14, 15, 16, 17, 16, 17, 18, 19, 20, 21, 20, 21, 22, 23, 24 25, 24, 25 26 27, 28, 29, 28 29, 30, 31, 32, 1.

P-BOX PERMUTATION (56 -> 48): 16, 7, 20, 21, 29, 12, 28, 17, 1, 15, 23, 26, 5, 18, 31, 10, 2, 8, 24, 14, 32, 27, 3, 9, 19, 13, 30, 6, 22, 11, 4, 25.

Data Encryption Standard (DES) / The function $f(R_{i-1}, K_i)$

S-BOX-1 to S-BOX-8 (6 -> 4): The fist two bits determine the row; the next four bits determines the column.

14, 4, 13, 1, 2, 15, 11, 8, 3, 10, 6, 12, 5, 9, 0, 7,
0, 15, 7, 4, 14, 2, 13, 1, 10, 6, 12, 11, 9, 5, 3, 8,
4, 1, 14, 8, 13, 6, 2, 11, 15, 12, 9, 7, 3, 10, 5, 0,
15, 12, 8, 2, 4, 9, 1, 7, 5, 11, 3, 14, 10, 0, 6, 13.

15, 1, 8, 14, 6, 11, 3, 4, 9, 7, 2, 13, 12, 0, 5, 10,
3, 13, 4, 7, 15, 2, 8, 14, 12, 0, 1, 10, 6, 9, 11, 5,
0, 14, 7, 11, 10, 4, 13, 1, 5, 8, 12, 6, 9, 3, 12, 15,
13, 8, 10, 1, 3, 15, 4, 2, 11, 6, 7, 12, 0, 5, 14, 9.

10, 0, 9, 14, 6, 3, 15, 5, 1, 13, 12, 7, 11, 4, 2, 8,
13, 7, 0, 9, 3, 4, 6, 10, 2, 8, 5, 14, 12, 11, 15, 1,
13, 6, 4, 9, 8, 15, 3, 0, 11, 1, 2, 12, 5, 10, 14, 7,
1, 10, 13, 0, 6, 9, 8, 7, 4, 15, 14, 3, 11, 5, 2, 12.

7, 13, 14, 3, 0, 6, 9, 10, 1, 2, 8, 5, 11, 12, 4, 15,
13, 8, 11, 5, 6, 15, 0, 3, 4, 7, 2, 12, 1, 10, 14, 9,
10, 6, 9, 0, 12, 11, 7, 13, 15, 1, 3, 14, 5, 2, 8, 4,
3, 15, 0, 6, 10, 1, 13, 8, 9, 4, 5, 11, 12, 7, 2, 14.

2, 12, 4, 1, 7, 10, 11, 6, 8, 5, 3, 15, 13, 0, 14, 9,
14, 11, 2, 12, 4, 7, 13, 1, 5, 0, 15, 10, 3, 9, 8, 6,
4, 2, 1, 11, 10, 13, 7, 8, 15, 9, 12, 5, 6, 3, 0, 14,
11, 8, 12, 7, 1, 14, 2, 13, 6, 15, 0, 9, 10, 4, 5, 3.

12, 1, 10, 15, 9, 2, 6, 8, 0, 13, 3, 4, 14, 7, 5, 11,
10, 15, 4, 2, 7, 12, 9, 5, 6, 1, 13, 14, 0, 11, 3, 8,
9, 14, 15, 5, 2, 8, 12, 3, 7, 0, 4, 10, 1, 13, 11, 6,
4, 3, 2, 12, 9, 5, 15, 10, 11, 14, 1, 7, 6, 0, 8, 13.

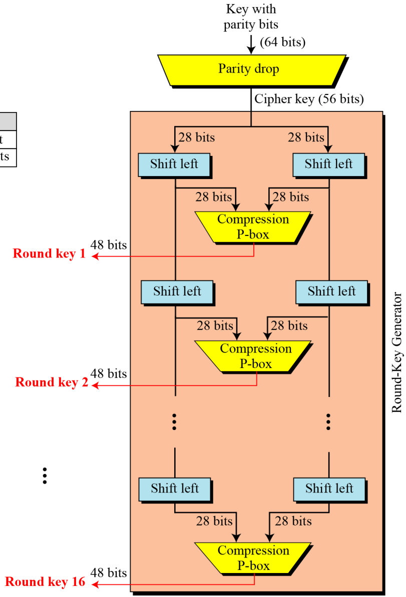
4, 11, 2, 14, 15, 0, 8, 13, 3, 12, 9, 7, 5, 10, 6, 1,
13, 0, 11, 7, 4, 9, 1, 10, 14, 3, 5, 12, 2, 15, 8, 6,
1, 4, 11, 13, 12, 3, 7, 14, 10, 15, 6, 8, 0, 5, 9, 2,
6, 11, 13, 8, 1, 4, 10, 7, 9, 5, 0, 15, 14, 2, 3, 12.

13, 2, 8, 4, 6, 15, 11, 1, 10, 9, 3, 14, 5, 0, 12, 7,
1, 15, 13, 8, 10, 3, 7, 4, 12, 5, 6, 11, 0, 14, 9, 2,
7, 11, 4, 1, 9, 12, 14, 2, 0, 6, 10, 13, 15, 3, 5, 8,
2, 1, 14, 7, 4, 10, 8, 13, 15, 12, 9, 0, 3, 5, 6, 11.

Data Encryption Standard (DES) / Key Scheduling

Shifting

Rounds	Shift
1, 2, 9, 16	one bit
Others	two bits



Data Encryption Standard (DES) / Key Scheduling

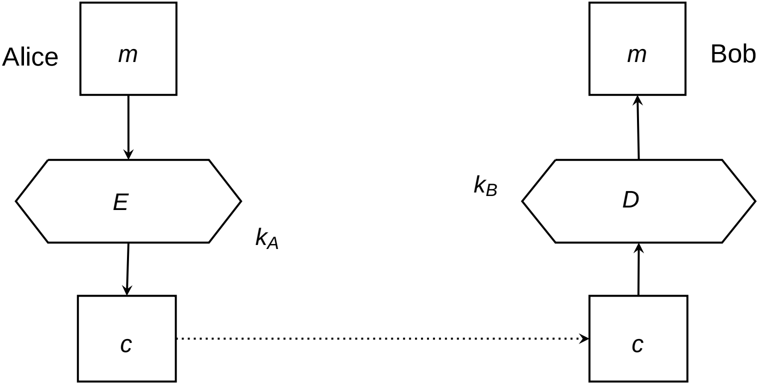
KEY PERMUTATION (64 -> 56): 57, 49, 41, 33, 25, 17, 9, 1, 58, 50, 42, 34, 26, 18, 10, 2, 59, 51, 43 35 27 19, 11, 3, 60, 52, 44, 36, 63, 55, 47 39, 31, 23, 15, 7, 62, 54, 46, 38, 30, 22, 1, 6, 61, 53, 45, 37, 29, 21, 13, 5, 28, 20, 12, 4.

KEY SHIFTS PER ROUND

Round	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
# of shifts	1	1	2	2	2	2	2	2	1	2	2	2	2	2	2	1

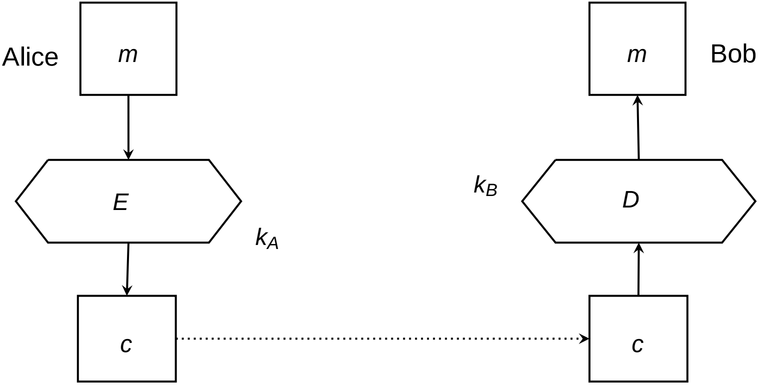
COMPRESSION PERMUTATION (56 -> 48): 14, 17, 11, 24, 1, 5, 3, 28, 15, 6, 21, 10, 23, 19, 12, 4, 26, 8, 16, 7, 27, 20, 13, 2, 41, 52, 31 37, 47, 55, 30, 40, 51, 45, 33, 48, 44, 49, 39 56, 34, 53, 46, 42, 50, 36, 29, 32.

The RSA algorithm



The RSA algorithm

1. Bob chooses primes p and q. Computes $n = p \cdot q$.



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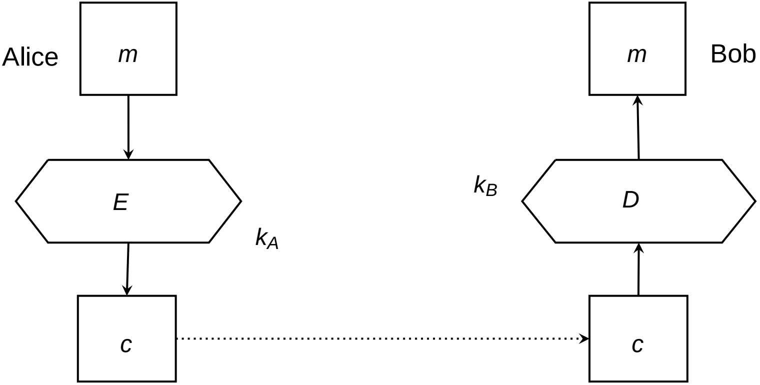
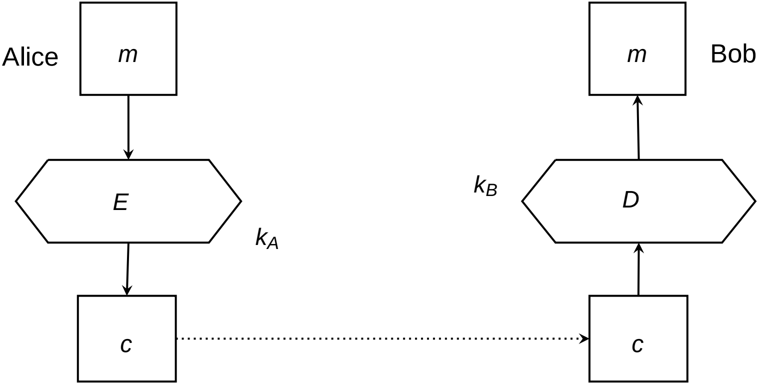
2. Bob chooses e with $\text{GCD}\big(e, (p-1)\cdot(q-1)\big) = 1$.



The RSA algorithm

1. Bob chooses primes p and q. Computes $n = p \cdot q$.
2. Bob chooses e with $\text{GCD}\Big(e, (p-1)\cdot(q-1)\Big) = 1$.
3. Bob computes d with $d \cdot e \equiv 1 \mod (p-1)\cdot(q-1)$.



The RSA algorithm

1. Bob chooses primes p and q. Computes $n = p \cdot q$.
2. Bob chooses e with $\mathrm{GCD}\Big(e, (p-1)\cdot(q-1)\Big) = 1$.
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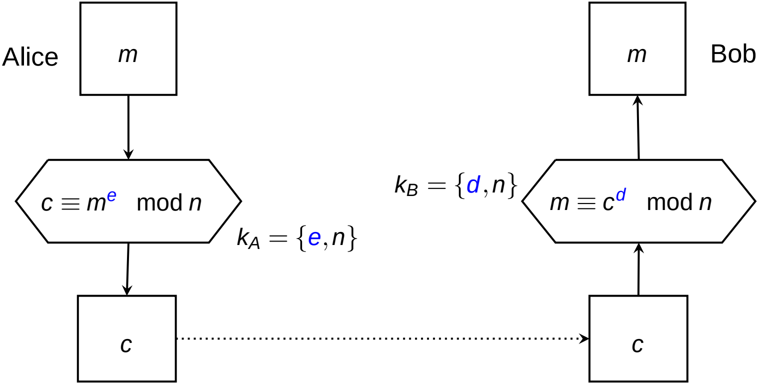

RSA in action

Alice wants to send a message m to Bob:

$$\text{Let } m = \text{“}\mathtt{\textbf{Hello}}\text{”} = \mathtt{\textbf{0x48656C6C6F}} = 310939249775.$$

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Alice gets $\{e, n\}$ from Bob, computes & sends the ciphertext c:

$$\begin{aligned} c \equiv m^e &\equiv 310939249775^{1644903229909} \quad \Big(\mathrm{mod}\ 2199099802013\Big). \\ &\equiv 858640968629 \quad (= \text{“}\mathtt{\textbf{Çéık\&}}\text{”}) \end{aligned}$$

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Bob receives the ciphertext c and decrypt it using {d, n}:

$$\begin{aligned} m \equiv c^d &\equiv 858640968629^{2055797390629} && \Big(\mathrm{mod}\ 2199099802013\Big). \\ &\equiv 310939249775 \quad (= \text{“}\mathtt{\textbf{Hello}}\text{”}) \end{aligned}$$

How does the RSA decryption works?

Definition (Euler's totient function)

Let n be an integer.

$$\phi(n) := \text{“The number of integers } 1 \leq a \leq n \text{ such that } \mathrm{GCD}(a,n) = 1\text{”}.$$

Lemma

$\phi(n) = \phi(p \cdot q) = (p-1) \cdot (q-1)$.

Theorem

If $GCD(a,n) = 1$ then $a^{\phi(n)} \equiv 1 \mod n$.

Now,

$$c^d \equiv (m^e)^d \equiv m^{1+k\cdot\phi(n)} \equiv m \cdot (m^{\phi(n)})^k \equiv m \cdot 1^k \equiv m \mod n.$$

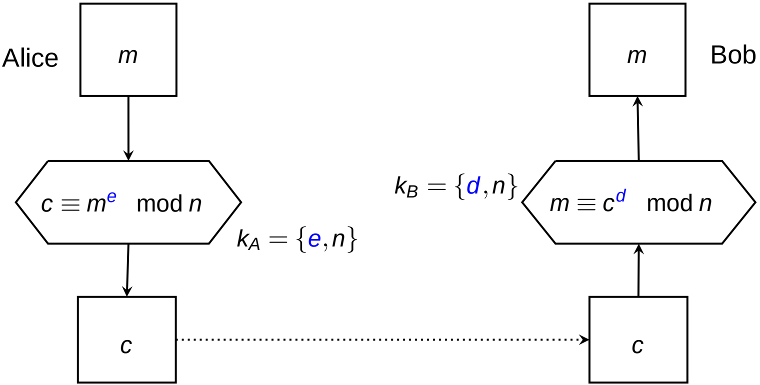
A 1024-bit RSA Key Pair

- $\{e, n\}$ is Bob's public key.
- $\{d, n\}$ is Bob's private key.
- A 1024-bit real life example for $\{e, n\}$ and $\{d, n\}$:

```
-----BEGIN RSA PUBLIC KEY-----
98903585440747599384191320255954189656318148122089021285657780860309099864821 41\
61076066778910536737051038839999777729455374045174487243350037733416639711850 53\
39293845960769712410984156894969469738678553933337641721313425918180516603240 62\
4522290105265565886483476797092062038811264788746288467833203265965 2219,
10438979421523677490638252295322065521654535721060522931311413897271600366022 68\
08705772017491398949753817944988638218003392833393273918097591973229650906151 49\
20362836849521069991467875040592817931791644012871146435291241331010484648733 53\
37914381455578220039854103376720743159149457332624961822653722962734 3777
-----END RSA PUBLIC KEY-----
```

```
-----BEGIN RSA PRIVATE KEY-----
60896885011141551638324626835139206081818916976938394877942378438363254397688 48\
96604426412160960361021198228627940644422432475043851974209073046926271643191 54\
62555051230485641077819927134910694147560629919427454813253574609201185666958 87\
6224525091700085797266395012286691829822826276550454575385878946349 8444,
10438979421523677490638252295322065521654535721060522931311413897271600366022 68\
08705772017491398949753817944988638218003392833393273918097591973229650906151 49\
20362836849521069991467875040592817931791644012871146435291241331010484648733 53\
37914381455578220039854103376720743159149457332624961822653722962734 3777
-----END RSA PRIVATE KEY-----
```

Threats against RSA

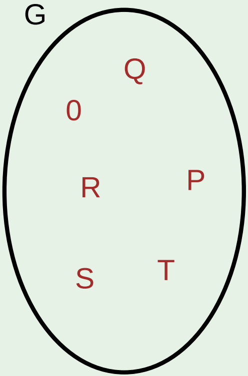


What can Eve do?

- Eve can intercept n, e, c.
- Eve does not know p, q, d.
- Eve cannot factor n. (assumption)

Group

A set G



A binary operation on G

- Q + R = T
- S + P = R
- 0 + S = S (identity)
- 0 + 0 = 0
- P + T = 0 (inverse)

Group

Definition

A group is a pair $(G,+)$ consisting of a nonempty set G and a binary operation $+$, (closed) on G, such that $(\forall\, P,Q,R \in G)$

- Binary operation is associative; $(P+Q)+R = P+(Q+R)$,
- A unique identity exists; $0+P = P+0 = P$,
- Every element has a unique inverse; $P+Q = Q+P = 0$.

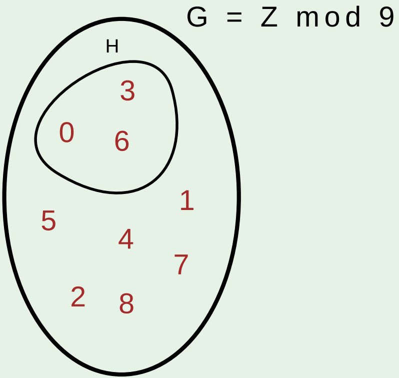
Furthermore, $(G,+)$ is abelian if $P+Q = Q+P \quad \forall\, P,Q \in G$.

Examples

- $\mathbb{Z}/p\mathbb{Z}$ is an abelian group. (Simply “mod p” arithmetic)
- An elliptic curve is a group. (We will define this later)

Subgroup

H is a subset of *G*



$H \subset G$

+	0	3	6
0	0	3	6
3	3	6	0
6	6	0	3

Check

- Closed
- Identity
- Inverses
- Associativity

Subgroup

Definition

A subset H of a group G which is

- closed under the binary operation of G,
- a group itself,

is called a subgroup of G. $(H \subseteq G)$

Cyclic (Sub)group, Generator

Definition

Let $P \in G$, then

$$H = \left\{ nP = \underbrace{P + P + \ldots + P}_{n \text{ times}} \mid n \in \mathbb{Z} \right\}$$

is the cyclic subgroup of G generated by P. $\quad \left(H = \langle P \rangle \right)$

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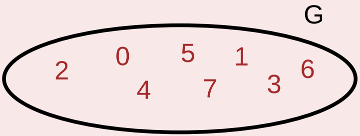
Remark

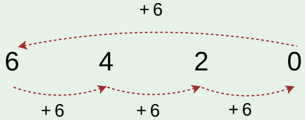
- If an element $P \in G$ generates G, then P is a generator for G. $\big(G = \langle P \rangle\big)$
- G is a cyclic group if there is some element $P \in G$ that generates G.
- The number of elements in $\langle P \rangle$ is called the order of P and is denoted by $|\langle P \rangle|$.

Cyclic (Sub)group, Generator

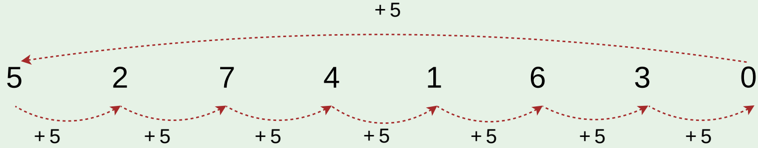
Consider integers modulo 8.



6 is not a generator for G.



5 is a generator for G. $G = \langle 5 \rangle$



Discrete Logarithm Problem

- Let $(G, +)$ be a cyclic group of order n and let P be a generator of G.
- Given $Q \in G$ find the unique k such that $0 \le k \le n-1$ and $Q = kP$.
- Finding k is called Discrete Logarithm Problem (DLP).
- The complexity of DLP depends on the selection of the group G.

Note: If the group is written multiplicatively, the notation is changed to $Q = P^k$.

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1. Either Alice or Bob picks a prime p and a generator α.
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7. Alice calculates the shared secret as $K \equiv (\alpha^y)^x \mod p$.
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- Though Eve may know p, α, $\alpha^x \mod p$ and $\alpha^y \mod p$,
- She cannot recover K
- Unless she solves the DLP and finds out either x or y.

DH in action

1. Alice picks a prime $p = 558494556463$ and a generator $\alpha = 197214177966$.

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8. Bob: $K \equiv (\alpha^x)^y \equiv 542167786127^{306801011233} \equiv 306801011233 \mod p$.

ElGamal

As usual Alice wants to send a message to Bob.

- Let $G = \langle P \rangle$ be a cyclic group.
- Bob's public key is $Q = kP$.
- Bob's private key is k.
- Plaintext is $M \in G$.

Alice performs:

ElGamal Encryption

input : Q, M.
output : $\{C_0, C_1\}$.

Select a random r, $0 < r < |\langle P \rangle|$.

Compute $C_0 = rP$.

Compute $C_1 = M + rQ$.

return $\{C_0, C_1\}$. (The ciphertext)

Bob performs:

ElGamal Decryption

input : k, $\{C_0, C_1\}$.
output : M.

Compute $M = C_1 - kC_0$.

return M.

ElGamal

How does the encryption works?

- We have the relation $Q = kP$.
- Encryption is $C_1 = (M + rQ), \quad C_0 = (rP)$.
- Decryption is $M = (C_1 - kC_0)$.
- So, decryption corresponds to

$$\begin{cases} C_1 - kC_0 = \\ (M + rQ) - k(rP) = \\ M + rQ - r(kP) = \\ M + rQ - rQ = \\ M \end{cases}$$

Elliptic Curves

Definition *(A simplified non-technical version)*

Let $p > 2$ be a prime. Let A, B be integers satisfying

$$0 \leq A < p, \qquad 0 \leq B < p, \qquad 4A^3 + 27B^3 \not\equiv 0 \mod p.$$

An elliptic curve is the set of points

$$E := \left\{ (x, y) \mid (0 \leq x < p) \text{ and } (0 \leq y < p) \text{ and } (y^2 \equiv x^3 + Ax + B \mod p) \right\}$$

together with a distinguished point $\mathscr{O}$ (the point at infinity).

- We have a set of points.
- Our goal is to form a group.
- All we need is a binary operation!

Elliptic Curves / The Group Law

Bezout's Theorem *(A simplified non-technical version)*

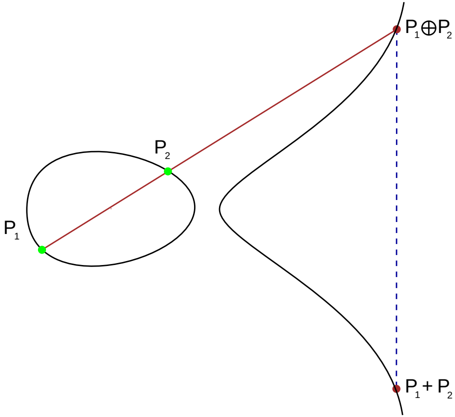
Two curves of degree m and n intersect in mn points.

Remark

An elliptic curve and a line intersect at 3 points.

Elliptic Curves / The Group Law

- We have a set of points.
- Our goal is to form a group.
- And the binary operation is:



With this binary operation;

- We select $\mathscr{O}$ as the identity element.
- The inverse of a point (x,y) is $(x,-y)$.

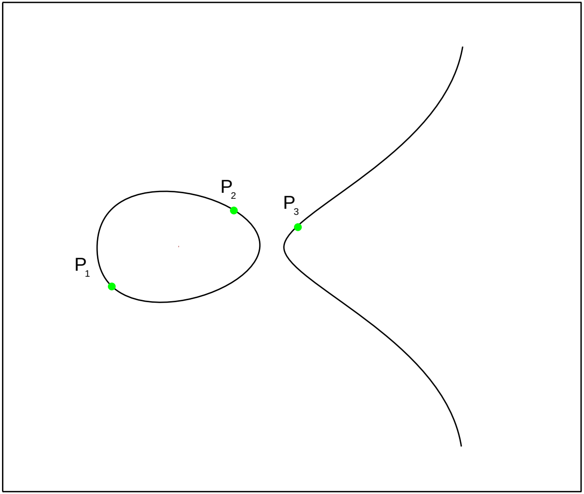
$$y^2 = x^3 + Ax + B$$

$$y = \pm\sqrt{x^3 + Ax + B}$$

- The only axiom to check is the associativity, i.e.

$$(P_1 + P_2) + P_3 = P_1 + (P_2 + P_3).$$

Elliptic Curves / The Group Law



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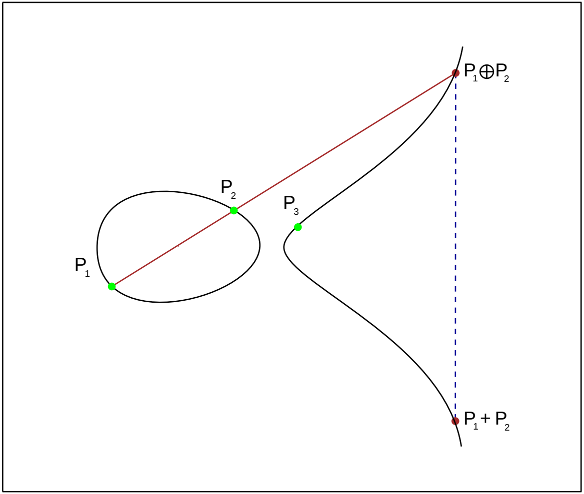

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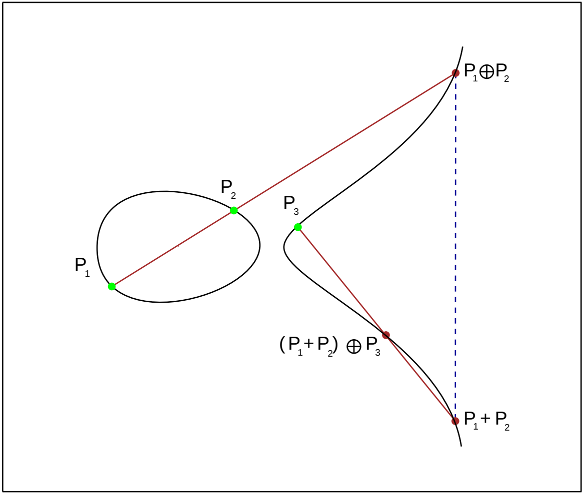

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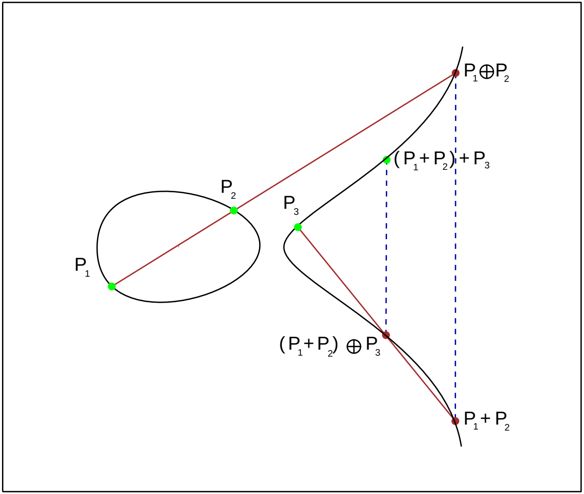

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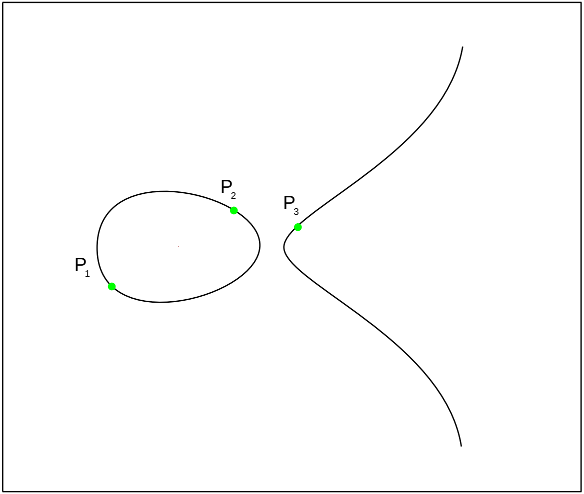

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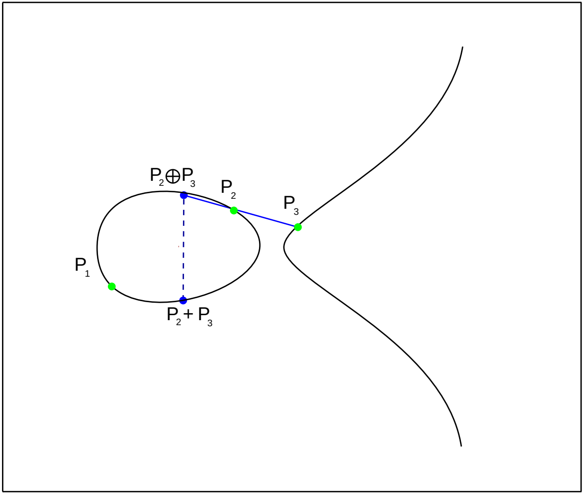

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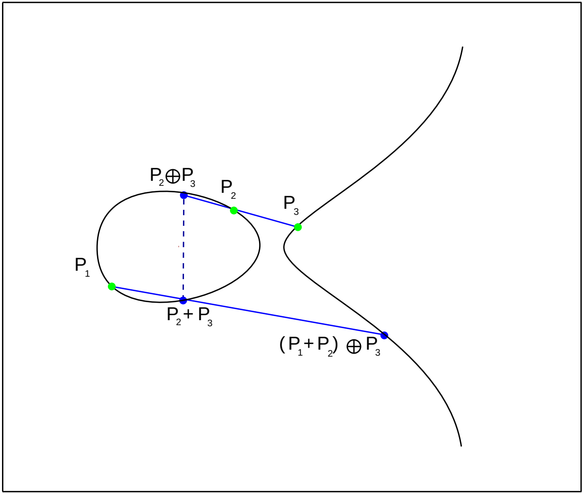

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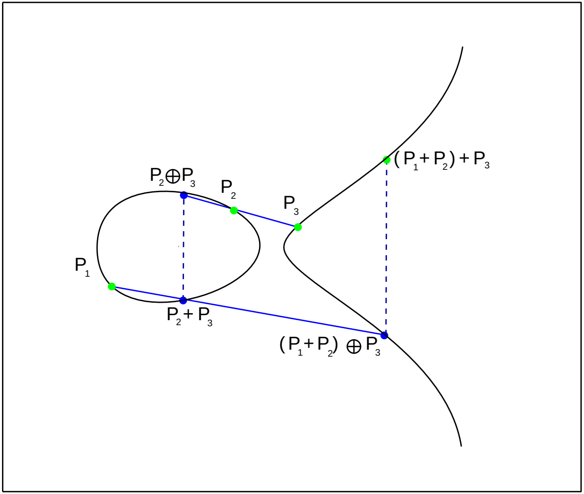

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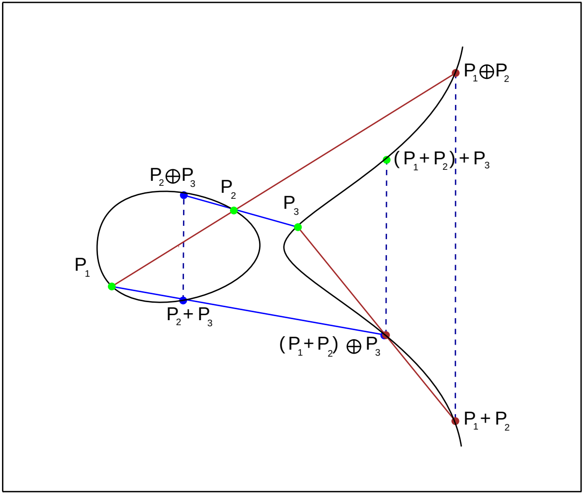
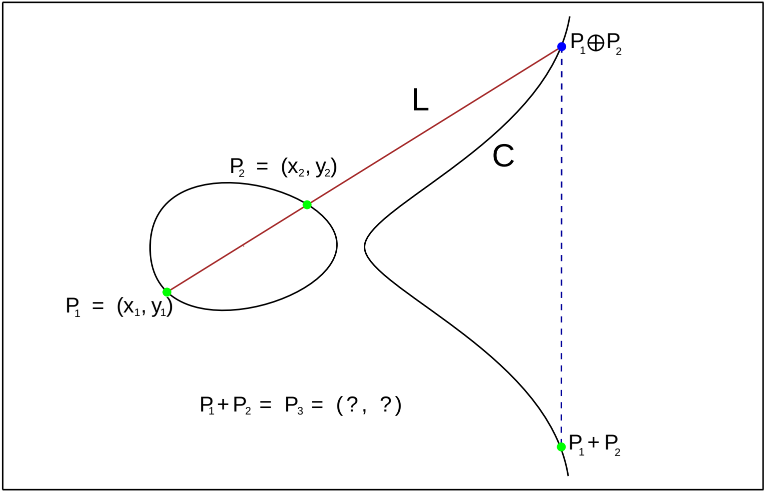


Elliptic Curves / Explicit Point Addition Formulae ($P_1 \neq P_2$)



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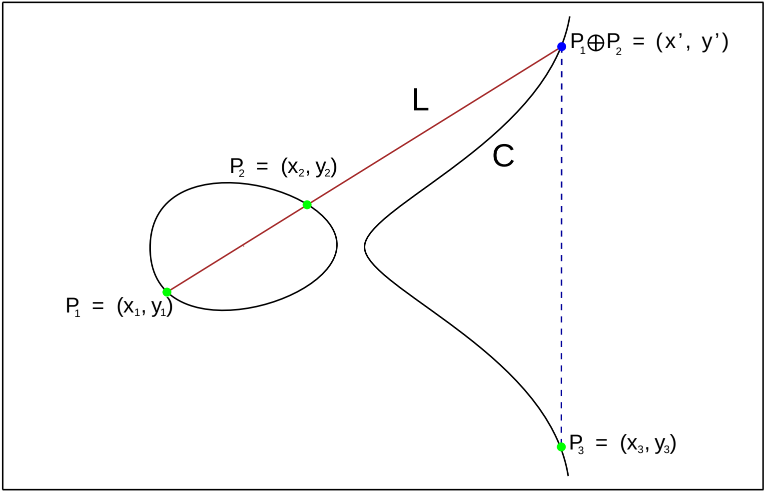
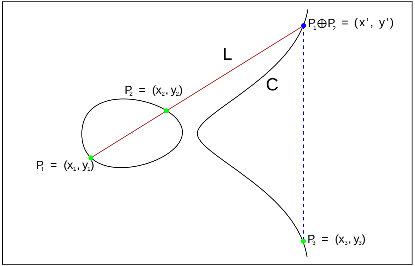


$\mathbf{L}: y = \lambda x + \beta$

where

$$\lambda = (y_2 - y_1)/(x_2 - x_1)$$

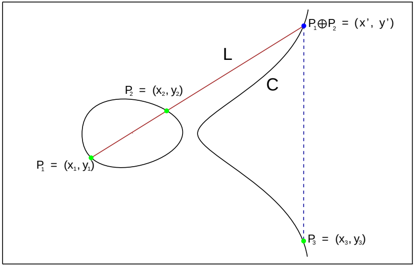


$\mathbf{C}: y^2 = x^3 + Ax + B \quad \longrightarrow \quad (x^3 + Ax + B - y^2) = (x - x_1)(x - x_2)(x - x')$

$x^3 + Ax + B - (\lambda x + \beta)^2 = x^3 - (x_1 + x_2 + x')x^2 + (x_1x_2 + x_2x' + x'x_1)x - (x_1x_2x')$

$x^3 - \lambda^2 x^2 + (A - 2\lambda\beta)x + (B - \beta^2) = x^3 - (x_1 + x_2 + x')x^2 + (x_1x_2 + x_2x' + x'x_1)x - (x_1x_2x')$

Elliptic Curves / Explicit Point Addition Formulae $(P_1 \neq P_2)$



L : $y = \lambda x + \beta$

where

$$\lambda = (y_2 - y_1)/(x_2 - x_1)$$

C : $y^2 = x^3 + Ax + B \quad \longrightarrow \quad (x^3 + Ax + B - y^2) = (x - x_1)(x - x_2)(x - x')$

$$x^3 + Ax + B - (\lambda x + \beta)^2 = x^3 - (x_1 + x_2 + x')x^2 + (x_1x_2 + x_2x' + x'x_1)x - (x_1x_2x')$$

$$x^3 - \lambda^2x^2 + (A - 2\lambda\beta)x + (B - \beta^2) = x^3 - (x_1 + x_2 + x')x^2 + (x_1x_2 + x_2x' + x'x_1)x - (x_1x_2x')$$

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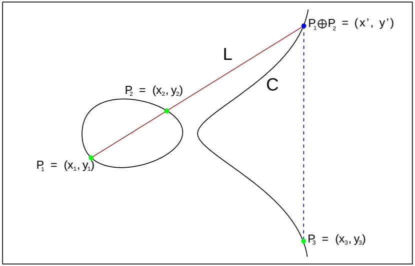

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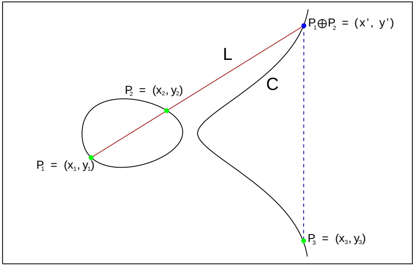
Elliptic Curves / Explicit Point Addition Formulae $(P_1 \neq P_2)$

$$\lambda^2 = x_1 + x_2 + x'$$

$$x' = \lambda^2 - x_1 - x_2$$

$$x_3 = x' = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)^2 - x_1 - x_2$$



$\mathbf{L} : y = \lambda x + \beta$

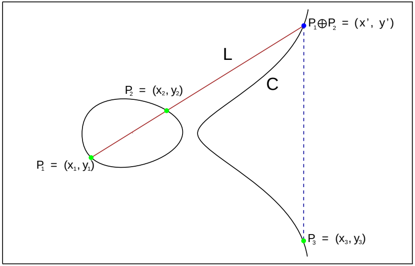
$$y_3 = -y' = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x_1 - x_3) - y_1$$

Elliptic Curves / Explicit Point Addition Formulae ($P_1 \neq P_2$)

$$\lambda^2 = x_1 + x_2 + x'$$

$$x' = \lambda^2 - x_1 - x_2$$

$$x_3 = x' = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)^2 - x_1 - x_2$$



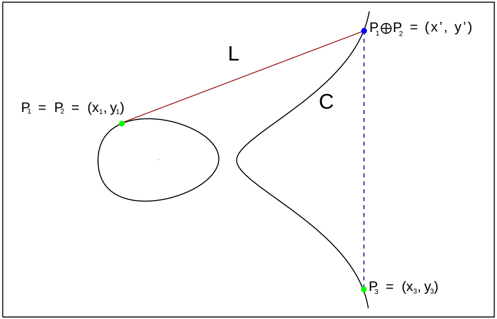
$\mathbf{L} : y = \lambda x + \beta$

$$y_3 = -y' = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x_1 - x_3) - y_1$$

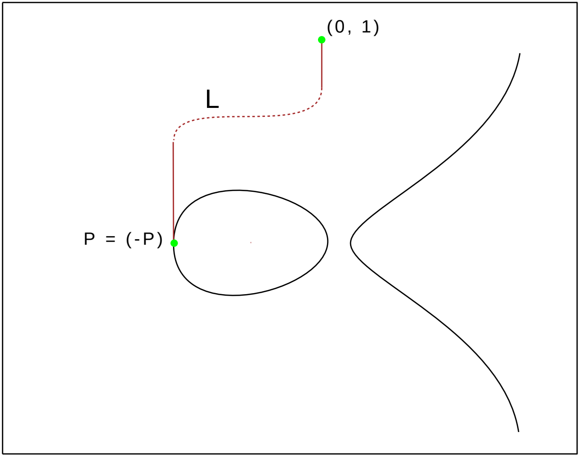
Elliptic Curves / Explicit Point Addition Formulae $(P_1 = P_2)$

$$x_3 = \left(\frac{3{x_1}^2 + A}{2y_1}\right)^2 - 2x_1$$

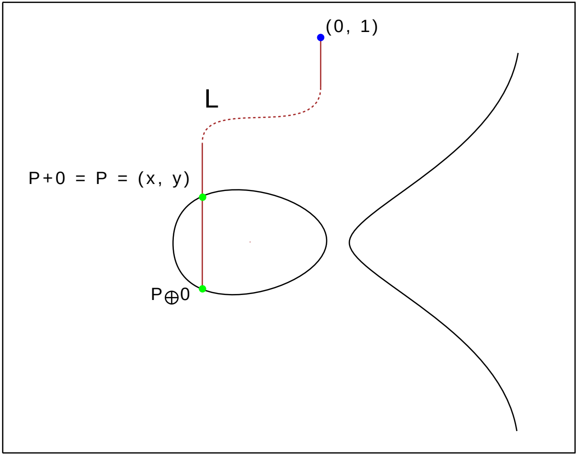
$$y_3 = \left(\frac{3{x_1}^2 + A}{2y_1}\right)(x_1 - x_3) - y_1$$



Elliptic Curves / Explicit Point Addition $(P = -P)$



Elliptic Curves / Explicit Point Addition ($P + \mathscr{O} = P$)



input : $P_1 = (x_1, y_1), P_2 = (x_2, y_2) \in E \mod p$.
output: $P_1 + P_2 = (x_3, y_3) \in E \mod p$.
if $P_1 = \mathcal{O}$ **then** **return** P_2.
else if $P_2 = \mathcal{O}$ **then** **return** P_1.
else if $x_1 = x_2$ **then**
    if $y_1 \neq y_2$ **then** **return** $\mathcal{O}$.
    else if $y_1 = 0$ **then** **return** $\mathcal{O}$.
    else
        $x_3 := ((3x_1^2 + a)/(2y_1))^2 - 2x_1 \mod p$.
        $y_3 := ((3x_1^2 + a)/(2y_1))(x_1 - x_3) - y_1 \mod p$.
        return (x_3, y_3).
    end
else
    $x_3 := ((y_1 - y_2)/(x_1 - x_2))^2 - x_1 - x_2 \mod p$.
    $y_3 := ((y_1 - y_2)/(x_1 - x_2))(x_1 - x_3) - y_1 \mod p$.
    return (x_3, y_3).
end

Elliptic Curves / A toy example

$E: y^2 = x^3 + 77x + 92 \mod 137.$

$4A^3 + 27B^3 \equiv 67 \not\equiv 0 \mod p$. So, E is an elliptic curve.

- $(x_1, y_1) = (95, 77) = P$ satisfies E.
- $(95, 77) + (95, 77) = (56, 31) = 2P$
- $(56, 31) + (95, 77) = (98, 67) = 3P$
- $(98, 67) + (95, 77) = (16, 25) = 4P$
- $\ldots$

ElGamal (Revisited)

As usual Alice wants to send a message to Bob.

- Let $G = \langle P \rangle$ be a cyclic group.
- Bob's public key is $Q = kP$.
- Bob's private key is k.
- Plaintext is $M \in G$.

Alice performs:

ElGamal Encryption

input : Q, M.
output : $\{C_0, C_1\}$.

Select a random r, $0 < r < |\langle P \rangle|$.

Compute $C_0 = rP$.

Compute $C_1 = M + rQ$.

return $\{C_0, C_1\}$. (The ciphertext)

Bob performs:

ElGamal Decryption

input : k, $\{C_0, C_1\}$.
output : M.

Compute $M = C_1 - kC_0$.

return M.

Thanks.